The Universal Covering Space of a Haken n–Manifold

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We define the class of Haken n–manifolds as a generalisation of Haken 3–manifolds. We prove that the interior of the universal covering of a Haken n–manifold is $\mathbf{R}^n$, which generalises a result of Waldhausen. The techniques used allow us to provide a new proof of Waldhausen's universal cover theorem for Haken 3–manifolds.

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1 Introduction

In the final section of Waldhausen's classic paper on Haken 3–manifolds [11], he proves that the interior of the universal covering space of a Haken 3–manifold is $\mathbf{R}^3$. A direct generalisation of his proof for Haken n–manifolds leads to some difficulties, which are discussed in Foozwell [5]. In this paper, we give a proof of a Waldhausen universal covering theorem in all dimensions, via induction on dimension on the manifold. Surprisingly, the approach in dimension three needs to be different, but we do obtain a new proof of the three-dimensional case that is similar in spirit to the higher dimensional proof.

In section 2, we give the basic definitions needed to define Haken n–manifolds. The definition is more complicated than the definition for 3–manifolds, and we use the boundary-pattern concept developed by Johannson.

In section 3, we present the proof of the main theorem for higher dimensional Haken manifolds, assuming that the result is true in dimension three.

In section 4, we give the proof of the main theorem in the three-dimensional case. We assume the result is true in dimension two, but this can be proved easily using the techniques from section 3, or just using the classical argument.

2 Preliminary definitions

Definition 2.1 Let M be a compact n–manifold with boundary, and let $\underline{\underline{m}}$ be a finite collection of compact, connected $(n-1)$–manifolds in ∂M. Let $i \in \{1, \ldots, n+1\}$. If the intersection of each collection i elements of $\underline{\underline{m}}$ is either empty or consists of $(n-i)$–manifolds, then $\underline{\underline{m}}$ is called a *boundary-pattern* for M.

Such a manifold is called a *manifold with boundary-pattern*. We use the notation $\left(M, \underline{\underline{m}}\right)$ when we wish to emphasise that M is a manifold with boundary-pattern. The elements of $\underline{\underline{m}}$ are called *faces* of the boundary-pattern.

We say that a boundary-pattern is *complete* if $\partial M = \bigcup \left\{A : A \in \underline{\underline{m}}\right\}$. If the boundary-pattern is not complete, a complete boundary-pattern can be formed by including the components of Closure $\left(\partial M \setminus \bigcup\{A : A \in \underline{\underline{m}}\}\right)$ together with the elements of $\underline{\underline{m}}$. This complete boundary-pattern is called the *completion* of $\underline{\underline{m}}$ and is denoted $\overline{\underline{\underline{m}}}$.

The *intersection complex* $K = K\left(\underline{\underline{m}}\right)$ of a manifold with boundary-pattern is

$$K = \bigcup_{A \in \underline{\underline{m}}} \partial A.$$

The intersection complex of a 3–manifold with boundary-pattern is a graph with vertices of degree three.

Definition 2.2 Suppose that $\left(M, \underline{\underline{m}}\right)$ and $\left(N, \underline{\underline{n}}\right)$ are manifolds with boundary-patterns. An *admissible map* between M and N is a continuous proper map $f\colon M \to N$ such that

$$\underline{\underline{m}} = \bigsqcup_{A \in \underline{\underline{n}}} \{B : B \text{ is a component of } f^{-1}(A)\}.$$

Furthermore, f must be transverse to the boundary-patterns.

Consider a disk with complete boundary-pattern consisting of i components. If $i = 1$, then such a disk has zero vertices, and we call such a disk a *zerogon*. For $i \geq 2$, such a disk has i vertices. A *bigon* has two vertices and a *triangle* has three vertices. Collectively, zerogons, bigons and triangles are called *small disks*[1].

[1] Monogons are disks with one vertex in the boundary. However, these do not play a role here, because they are not manifolds with boundary-pattern. Zerogons were called monogons in Foozwell [5].

Definition 2.3 Let K be the intersection complex of an n–manifold $\left(M, \underline{\underline{m}}\right)$. Suppose that for each admissible map $f\colon\ \left(\Delta, \underline{\underline{\delta}}\right) \to \left(M, \underline{\underline{m}}\right)$ of a small disk, there is a map $g\colon\ \Delta \to \partial M$, homotopic to f rel $\partial\Delta$, such that $g^{-1}(K)$ is the cone on $g^{-1}(K) \cap \partial\Delta$. Then the boundary-pattern $\underline{\underline{m}}$ of M is called a *useful boundary-pattern*.

Definition 2.4 Let $(J, \underline{\underline{j}})$ be a compact one-dimensional manifold with boundary-pattern and let $(M, \underline{\underline{m}})$ be an n–dimensional manifold with boundary-pattern. An admissible map $\sigma\colon\ (J, \underline{\underline{j}}) \to (M, \underline{\underline{m}})$ is an *inessential* curve if there is a disk $(\Delta, \underline{\underline{\delta}})$ and an admissible map $g\colon\ (\Delta, \underline{\underline{\delta}}) \to (M, \underline{\underline{m}})$ such that:

(1) $J = \mathrm{Closure}\left(\partial\Delta \setminus \bigcup\{A : A \in \underline{\underline{\delta}}\}\right)$,

(2) the completion $\left(\Delta, \overline{\underline{\underline{\delta}}}\right)$ is a small disk,

(3) $g|_J = \sigma$.

Otherwise σ is called an *essential* curve.

Definition 2.5 An admissible map $\varphi\colon\ (F, \underline{\underline{f}}) \to (M, \underline{\underline{m}})$ is called *essential* if for each essential curve $\sigma\colon\ (J, \underline{\underline{j}}) \to (F, \underline{\underline{f}})$ the composition $\varphi \circ \sigma\colon\ (J, \underline{\underline{j}}) \to (M, \underline{\underline{m}})$ is also essential. In particular, an essential submanifold F of M is a submanifold such that the inclusion map is essential.

Definition 2.6 A *Haken* 1*–cell* is an arc with complete (and useful) boundary-pattern. For $n > 1$, if $(M, \underline{\underline{m}})$ is an n–cell with complete and useful boundary-pattern and each face $A \in \underline{\underline{m}}$ is a Haken $(n-1)$–cell, then $(M, \underline{\underline{m}})$ is a *Haken n–cell*.

Thus a Haken 1–cell is of the form $([a, b], \{a, b\})$ for $a, b \in \mathbf{R}$. A Haken 2–cell is a disk with at least four sides.

Definition 2.7 Let $(M, \underline{\underline{m}})$ be a compact n–manifold with boundary-pattern. Let F be a codimension-one, properly embedded, two-sided, essential submanifold of M. A boundary-pattern is *induced* on F if it is obtained by taking all the intersections of ∂F with the elements of the boundary pattern $\underline{\underline{m}}$ on M. Equivalently, the boundary-pattern $\underline{\underline{f}}$ is the induced pattern if the inclusion $(F, \underline{\underline{f}}) \to (M, \underline{\underline{m}})$ is an admissible map.

Definition 2.8 Let $(M, \underline{\underline{m}})$ be a compact n–manifold with complete and useful boundary-pattern. Let F be a codimension-one, properly embedded, two-sided, essential submanifold of M whose boundary-pattern is induced from the boundary pattern on M and is complete and useful. Suppose that F is not admissibly boundary-parallel. Then the pair (M, F) is called a *good pair*.

Suppose that $(M, \underline{\underline{m}})$ is a manifold with boundary-pattern and that $(F, \underline{\underline{f}})$ is a codimension-one submanifold. Let N be the manifold obtained by splitting M open along F. There is an obvious map $q\colon\ N \to M$ that glues parts of the boundary of N together to regain M. We define a boundary-pattern $\underline{\underline{n}}$ on N by

$$B \in \underline{\underline{n}} \text{ if and only if } B \text{ is a component of } q^{-1}(A)$$

for $A \in \underline{\underline{m}}$ or $A = F$. This is the boundary-pattern that we will use whenever splitting situations arise.

Definition 2.9 A finite sequence

$$(M_0, F_0), (M_1, F_1), \ldots, (M_k, F_k)$$

of good pairs is called a *hierarchy of length k* for M_0 if the following conditions are satisfied:

(1) M_{i+1} is obtained by splitting M_i open along F_i and,

(2) M_{k+1} is a finite disjoint union of Haken n–cells.

A manifold with a hierarchy is called a *Haken n–manifold*. A Haken n–cell is a Haken manifold with a hierarchy of length zero.

We regard two Haken n–manifolds M and N as equivalent if there is an admissible homeomorphism $\varphi\colon\ \left(M, \underline{\underline{m}}\right) \to \left(N, \underline{\underline{n}}\right)$.

3 The main result

In proving our main theorem, we will use the following result of Doyle [3].

Theorem 3.1 *If P is a manifold with interior homeomorphic to $\mathbf{R}^n$ and boundary homeomorphic to $\mathbf{R}^{n-1}$, then P is homeomorphic to $\mathbf{R}^{n-1} \times [0, \infty)$, provided $n \neq 3$.*

Examples showing that ruling out three-dimensional manifolds is necessary are well known. The paper of Fox and Artin [7] is a pleasant way to discover such examples. We will also make use of the following folklore lemma, which in essence is the idea in Stallings [9].

Lemma 3.2 *Let P be an n–manifold that can be written as a countable union of compact subsets. Suppose that for each compact subset X in P, there is an embedding f of the standard open ball B into P such that $X \subset f(B)$. Then P is homeomorphic to $\mathbf{R}^n$.*

Theorem 3.3 *Let $\widetilde{M}$ be the universal cover of a Haken n–manifold M. Then the interior of $\widetilde{M}$ is homeomorphic to $\mathbf{R}^n$.*

Proof We have only defined boundary-patterns for compact manifolds. We will extend the definition to non-compact manifolds, in the case that we have a covering space. If $p\colon \widetilde{M} \to M$ is a covering map of a manifold with boundary-pattern $(M, \underline{\underline{m}})$, then we define a boundary pattern $p^{-1}(\underline{\underline{m}})$ for $\widetilde{M}$ by

$$p^{-1}(\underline{\underline{m}}) = \bigsqcup_{A \in \underline{\underline{m}}} \{B : B \text{ is a component of } p^{-1}(A)\}.$$

Let the sequence

$$(M_0, F_0), (M_1, F_1), \ldots, (M_k, F_k)$$

be a hierarchy for $M_0 = M$. To simplify notation let $N = M_1$ and $F = F_0$. We assume that the interior of the universal cover $\widetilde{N}$ of N is homeomorphic to $\mathbf{R}^n$. If $p\colon \widetilde{M} \to M$ is the covering projection, then the closure of each component of $\widetilde{M} \setminus p^{-1}(F)$ is homeomorphic to $\widetilde{N}$. There are a countable collection of such pieces and we label them $\{N_1, N_2, N_3, \ldots\}$.

Assume that each component of $p^{-1}(F)$ has interior homeomorphic to $\mathbf{R}^{n-1}$. There are countably many pieces of $p^{-1}(F)$, which we label as $\{F_1, F_2, F_3, \ldots\}$. We arrange the labelling so that $N^1 = N_1$,

$$N^i \cap N_{i+1} = F_i,$$

and

$$N^{i+1} = N^i \cup N_{i+1}.$$

We first aim to prove that $\text{Interior}(N^i) \cong \mathbf{R}^n$ for each $i \geq 1$. First observe that, by assumption, $\text{Interior}(N^1) \cong \mathbf{R}^n$. We assume that $\text{Interior}(N^j) \cong \mathbf{R}^n$ is true for all $j \leq i$ and then prove that $\text{Interior}(N^{i+1}) \cong \mathbf{R}^n$. Recall that $N^{i+1} = N^i \cup N_{i+1}$. Let $P = \text{Interior}(N^i) \cup \text{Interior}(F_i)$ and let $Q = \text{Interior}(F_i) \cup \text{Interior}(N_{i+1})$. Both P and Q are manifolds with interior homeomorphic to $\mathbf{R}^n$ and boundary homeomorphic to $\mathbf{R}^{n-1}$. By Doyle's theorem 3.1, it follows that each of P and Q are homeomorphic to $\mathbf{R}^{n-1} \times [0, \infty)$, provided we assume that $n > 3$.

Let us regard $P \cup Q$ as being formed by attaching a collar $Q = \mathbf{R}^{n-1} \times [0, \infty)$ of the boundary of P to ∂P. Thus $P \cup Q \cong \text{Interior}(P) \cong \mathbf{R}^n$. So $\text{Interior}(N^{i+1}) \cong \mathbf{R}^n$ as we aimed to prove.

Let X be a compact subset of $\text{Interior}(\widetilde{M})$. Then $X \subset N^i$ for some integer i. Since $\text{Interior}(N^i) \cong \mathbf{R}^n$, it follows that there is an open ball in $\text{Interior}(N^i)$ containing

X. Hence, there is an open ball in $\mathrm{Interior}(\widetilde{M})$ containing X, which shows that $\mathrm{Interior}(\widetilde{M}) \cong \mathbf{R}^n$. $\square$

4 The three-dimensional case

We use the following theorem of Doyle and Hocking [4] in this section.

Theorem 4.1 *Let M be a 3–manifold such that $\mathrm{Interior}(M) \cong \mathbf{R}^3$ and $\partial M \cong \mathbf{R}^2$. Suppose that $M \neq \mathbf{R}^2 \times [0,\infty)$. Then there is a polygonal graph $\Gamma \subset M$ such that $\Gamma \cap M$ is a point x and there is no closed 3-cell C in M containing Γ for which $\Gamma \setminus \{x\} \subset \mathrm{Interior}(C)$.*

We use theorem 4.1 as follows: suppose Y is a manifold with interior homeomorphic to $\mathbf{R}^3$ and boundary homeomorphic to $\mathbf{R}^2$, such that every graph $\Gamma \subset Y$ that meets ∂Y in a point can be contained in a ball that meets ∂Y in a disk, then Y is homeomorphic to $\mathbf{R}^2 \times [0,\infty)$. We will refer to such a ball as an *engulfing ball* for Γ.

Theorem 4.2 *Let M be an orientable Haken 3–manifold. Then the interior of the universal cover $\widetilde{M}$ of M is homeomorphic to $\mathbf{R}^n$.*

Proof Let us first suppose that M is closed. A result of Aitchison and Rubinstein [1] says that M has a very short hierarchy:

$$(M,F),(N,S),(P,D),$$

where F is a maximal collection if closed incompressible surfaces, S is a collection of spanning surfaces, P is a disjoint union of handlebodies and D is a collection of meridian disks in each handlebody.

Lemma 4.3 *Let P_i be a component of the universal covering space of a component of P, and let E be the closed unit ball in $\mathbf{R}^3$. There is an embedding $e\colon P_i \to E$ such that $\mathrm{Interior}(E) \subset e(P_i)$, and for each $A \in \underline{\underline{\widetilde{p_i}}}$, the closure of $e(A)$ in ∂E is a disk.*

Proof If P_i covers a solid torus, then P_i is homeomorphic to $D^2 \times \mathbf{R}$, which embeds in the unit ball as $E \setminus \{(0,0,\pm 1)\}$. If $A \in \underline{\underline{\widetilde{p_i}}}$, then $e(A)$ is bounded by lines that cover circles in the graph of $\underline{\underline{p}}$. Each such line has one end at $(0,0,1)$ and the other at $(0,0,-1)$, so the closure of $e(A)$ in ∂E is a disk.

If P_i covers a handlebody of positive genus, then we can visualise P_i as the regular neighbourhood in $\mathbf{H}^3$ of a graph in $\mathbf{H}^2$. Each vertex of the graph has degree four. The graph meets the boundary of $\mathbf{H}^2$ in a Cantor set. We shall refer to the points in this set as *Cantor points*. The closure of the regular neighbourhood of the graph is a ball. Clearly, there is an embedding $e\colon P_i \to E$ into the unit ball. We may choose e so that the Cantor points lie in the equator of E. We must show that if A is in the boundary-pattern of P_i, then the closure of $e(A)$ in the ball is a disk. Note that if L is a line in the boundary of $e(A)$, then the different ends of L must lie in different Cantor points. Thus, the closure of $e(A)$ is a disk. $\square$

Note that lemma implies that if T is an element of the boundary-pattern of $\widetilde{P}$, then $\mathrm{Interior}(\widetilde{P}) \cup \mathrm{Interior}(T)$ is homeomorphic to $\mathbf{R}^2 \times [0,\infty)$.

Lemma 4.4 *Let $\pi\colon \widetilde{N} \to N$ be the universal covering of a component of N, and let A be a component of $\pi^{-1}(\partial N)$. Then* $\mathrm{Interior}(\widetilde{N}) \cup A$ *is homeomorphic to* $\mathbf{R}^2 \times [0,\infty)$.

Proof Let $S_1, S_2, \ldots$ denote the components of $\pi^{-1}(S)$. Each S_i is homeomorphic to the universal cover $\widetilde{S}$ of S. The closure of each component of $\widetilde{N} \setminus \pi^{-1}(S)$ is homeomorphic to $\widetilde{P}$. We define collections $P_1, P_2, P_3, \ldots$ and $P^1, P^2, P^3, \ldots$ of submanifolds of $\widetilde{N}$ that satisfy the following:

(1) The collection $\{P_i\}$ covers $\widetilde{N}$. That is $\widetilde{N} = \bigcup_{i=1}^{\infty} P_i$.

(2) Each P_i is the closure of a component of $\widetilde{N} \setminus \pi^{-1}(S)$.

(3) The labelling is arranged so that

$$\begin{aligned} P^1 &= P_1, \\ P^i \cap P_{i+1} &= S_i, \text{ and} \\ P_{i+1} &= P^i \cup P_{i+1}. \end{aligned}$$

We aim to show that $V := \mathrm{Interior}(\widetilde{N}) \cup A$ is homeomorphic to $\mathbf{R}^2 \times [0,\infty)$, where A is a component of $\pi^{-1}(\partial N)$. Let Γ be a compact graph in V such that $\Gamma \cap \partial V$ is a point v. We will show that there is a ball $B \subset V$ such that $\Gamma \subset B$ and $B \cap \partial V$ is a disk containing v.

Since Γ is compact, there is some P^i containing Γ. Therefore, we will show that if $\Gamma \subset P^i$, then there is a ball $B \subset P^i$ such that $\Gamma \subset B$ and $B \cap (A \cap P^i)$ is a disk. We prove this by induction on the index i of the collection $\{P^i\}$. However, we need to prove a stronger statement, and to do this we need to define a boundary-pattern $\underline{\underline{p^i}}$ inductively for each P^i. Since $P^1 = P_1$, we define $\underline{\underline{p^1}} = \underline{\underline{p_1}}$. For $i > 1$ suppose we

have $A_1 \in \underline{\underline{p}}^i, A_2 \in \underline{\underline{p_{i+1}}}$ and that $A_1, A_2 \subset \partial P^i$. If $A_1 \cap A_2$ is an arc, the $A_1 \cap A_2 \in \underline{\underline{p}}^{i+1}$. If $A_1 \cap A_2$ is not an arc, then A_1 and A_2 belong to distinct elements of $\underline{\underline{p}}^{i+1}$.

We need to prove that if Γ is a compact graph in P^i that has non-empty intersection with finitely many faces of $\underline{\underline{p}}^i$, then there is a compact ball $B \subset P^i$ containing Γ and satisfying the following *face intersection conditions:*

- for each $A \in \underline{\underline{p}}^i$, if $\Gamma \cap A \neq \varnothing$, then $B \cap A$ is a disk,
- for each $A \in \underline{\underline{p}}^i$, if $\Gamma \cap A = \varnothing$, then $B \cap A = \varnothing$.

If $\Gamma \subset P^1$, then there is a ball in P^1 containing Γ and satisfying the face intersection conditions, because P^1 is the universal cover of a handlebody. We assume that the result is true for graphs in P^i. Let $\Gamma \subset P^{i+1}$. Then $\Gamma_2 = \Gamma \cap P_{i+1}$ is a graph in the universal cover of a handlebody, so there is a ball $B_2 \subset P_{i+1}$ that contains Γ_2 and satisfies the face intersection conditions. Let $\Gamma_1 = \Gamma \cap P^i$. Then, by assumption, there is a ball $B_1 \subset P^i$ that contains Γ_1 and satisfies the face intersection conditions. In particular, both $B_1 \cap S_i$ and $B_2 \cap S_i$ are disks.

If $B_1 \cap B_2$ is a disk, then $B_1 \cup B_2$ is the ball we need. If $B_1 \cap B_2$ is not a disk, then we need to modify at least one of B_1 or B_2.

Observe that $B_1 \cap B_2$ is a compact subset of Interior$(P^1 \cap P_{i+1}) =$ Interior$(S_i) \cong \mathbf{R}^2$, so there is a disk in Interior(S_i) containing $B_1 \cap B_2$. Let U be a sufficiently small bicollar of this disk. Then $B_1 \cup (U \cap P^i)$ is a ball and so is $B_2 \cup (U \cap P_{i+1})$. Now $B_1 \cup B_2 \cup U$ is a ball in P^{i+1} that contains Γ and satisfies the face intersection conditions for P^{i+1}.

So we have proved: if Γ is a compact graph in P^i that has non-empty intersection with finitely many faces of $\underline{\underline{p}}^i$, then there is a compact ball $B \subset P^i$ that contains Γ and satisfies the face intersection conditions. In particular, if $\Gamma \subset V$ such that $\Gamma \cap \partial V$ is a point, then there is a ball $B \subset V$ containing Γ such that $B \cap \partial V$ is a disk. Then theorem 4.1 says that V is homeomorphic to $\mathbf{R}^2 \times [0, \infty)$. $\square$

To show that $\widetilde{M}$ is homeomorphic to $\mathbf{R}^3$, we repeat the above argument. However, the details are less complicated than above, because if F is a closed surface, then its universal cover is $\mathbf{R}^2$ (rather than a missing boundary plane). $\square$

5 Conclusion

The universal covering space result establishes Haken n–manifolds as a special class of spaces worthy of further study. Mike Davis [2], for example, has produced examples of

aspherical 4–manifolds with universal covering spaces not homeomorphic to $\mathbf{R}^n$. We have shown elsewhere [5], using a direct generalisation of Waldhausen's proof in [10], that the word problem is solvable for the fundamental group of a Haken n–manifold.

Probably the most important open problem at the moment is the question of topological rigidity for Haken n–manifolds.

Question If $(M, \underline{\underline{m}})$ and $(N, \underline{\underline{n}})$ are Haken n–manifolds, that are admissibly homotopy equivalent, are they homeomorphic?

In particular, answering this question in dimension four would be of great interest. The techniques of Waldhausen [11] do not appear to be directly generalisable to the situation in higher dimensions. It seems that a new approach is required.

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